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IMPLEMENTING TENSOR ANALYSIS IN *Mathematica* WITH ILLUSTRATIONS FROM SCHWARZCHILD GRAVITATION

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Expressions that play roles in tensor analysis, such as Christofell symbols and curvature tensors, are coded as *Mathematica* modules by straightforward transcription of their defining equations. The built-in functions can then be used to perform tensor analysis and numerical valuations. The utility of the *Mathematica* formulation is illustrated by examples from Schwarzchild gravitation.

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INTRODUCTION

Andrew and Fleming (ref 1) discussed the calculation of null geodesics for the Schwarzchild, Kerr-Newman, and Winicour metrics. Numerical computation employed FORTRAN code, which was produced by *Mathematica* (ref 2). The *Mathematica* code employed to generate the FORTRAN code for the neutral (Q = 0) Kerr-Newman geodesics was presented. Reference 1 provides a convincing demonstration of the utility of symbolic manipulation software for the development of error-free, complex FORTRAN (or C) code.

In this report, a general approach to problems in tensor analysis (ref 3) employing *Mathematica* is described. The standard expressions of tensor calculus are transcribed directly into *Mathematica* modules in Part I. The operation of the modules is illustrated as they are described by application to a simple two-dimensional curved space. The present formulation allows enumeration and simplification of complex tensor equations, employing built-in *Mathematica* functions (such as Expand[], Together[], or Simplify[]).

The code is applied to address some simple problems pertinent to Schwarzchild space-time in Part II. Employing the built-in *Mathematica* operator Simplify[], the modules described produced FORTRAN coded Runge-Kutta geodesic equations for Kerr-Newman space-time in less than thirty minutes on a 386-based PC operating at 30 MHz. Of course, if simplification based on more efficient *Mathematica* operations (e.g., combinations of Apart[], Expand[], and Together[]) can be achieved, symbolic computation times can be dramatically reduced. Computation times are much shorter for the Schwarzchild geodesic equations.

PART I. TENSOR ANALYSIS IN *Mathematica*

Covariant Metric Tensor

The starting point for a typical problem in tensor analysis is the specification of the (covariant) metric tensor.

The operation of the *Mathematica* tensor analysis modules defined here is demonstrated by applications to geometry on the surface of a helicoid in R^3 having

$$g = \|g_{ij}\| = \begin{pmatrix} 1 & 0 \\ 0 & c^2+x[1][s]^2 \end{pmatrix}$$

where the coordinates are x[1][s] and x[2][s] and c is constant. This g is referred to as the "example metric" in Part I. One encodes this g directly into *Mathematica* via

```
g=DiagonalMatrix[{1,c^2 + x[1][s]^2}];
```

Note that coordinates are given in the form x[i][s]. For the current analysis, one could simply use x[i]. However, as in Part II, one is often interested in derivatives with respect to the line element s, and we choose to include this dependence from the beginning.

Christoffel Symbols

The first step in a tensor problem is the computation of the Christoffel symbols. Christoffel symbols of the first kind are defined as

$$\Gamma_{i\alpha\beta} = \Gamma_{i\beta\alpha} = \frac{1}{2}\left(\frac{\partial g_{ai}}{\partial x^{\beta}} + \frac{\partial g_{i\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{i}}\right)$$

Christoffel symbols of the second kind are given by

$$\Gamma^{\partial}_{\alpha\beta} = \sum_{i} g^{\delta i}\Gamma_{i\alpha\beta} = g^{\delta i}\Gamma_{i\alpha\beta}$$

where the last equality introduces "summation convention" which is adopted in this report and the contravariant metric tensor g^{ij} is the inverse of the covariant metric, i.e.,

$$\delta^{\alpha}_{\beta} = g^{\alpha i} g_{i\beta}$$

The following *Mathematica* module defines the contravariant metric tensor and Christoffel symbols for arbitrary metric tensors in spaces of arbitrary dimension.

```
setup[g_]:=Module[{dim=Length[g]},
        ginv    =Together[Inverse[g]];
        GAMMA       =Array[g1,{dim,dim,dim}];
        gamma   =Array[g2,{dim,dim,dim}];
Do[ Do[ Do[GAMMA[[i,k,j]]=GAMMA[[i,j,k]]=
        Together[(1/2)(D[g[[i,j]],x[k][s]]+
                D[g[[k,i]],x[j][s]]-D[g[[j,k]],x[i][s]])],
                    {i,dim}],{k,j,dim}],{j,dim}];
Do[ Do[ Do[gamma[[i,k,j]]=gamma[[i,j,k]]=
        Together[Sum[ginv[[i,l]]GAMMA[[l,j,k]],{l,dim}]],
                    {i,dim}],{k,j,dim}],{j,dim}]   ]
```

N. B., the partial derivative of a function f[u,v,...,x,...]with respect to the variable **x** is computed via D[f[u,v,...,x,...],x]in *Mathematica*. N.B., the *Mathematica* Together[] operator was effective in simplifying the resulting expressions for a number of "simple problems." Of course, the user could apply an other built-in (e.g., Simplify[]) or user-defined *Mathematica* operator.

After setup[g] is run, the contravariant metric tensor is returned as ginv, Christoffel symbols of the first kind are in the array *GAMMA*, i.e., Γ_{ijk} = *GAMMA*[[*i,j,k*]], and Christoffel symbols of the second kind are in the array gamma, i.e., $\Gamma^{i}_{j\,k}$ = *gamma*[[*i,j,k*]].

To compute the Christoffel symbols and g^{-1} for the test metric, enter

```
setup[g].
```

Example: Display gamma for the example metric,

```
gamma
```

returns

```
                             x[1][s]          x[1][s]
{{{0, 0}, {0, -x[1][s]}},{{0, -------------}, {---------------, 0}}},
                             2        2   2         2
                            c  + x[1][s]   c  + x[1][s]

{{{0, 0}, {0, -x[1][s]}},
```

Exercise: Code a *Mathematica* module to compute covariant derivatives.

The Curvature Tensor

The curvature tensor can be computed directly from the $\Gamma^i_{\alpha\,\beta}$

$$B^i_{\alpha jk} = \frac{\partial\Gamma^i_{\alpha j}}{\partial x^k} - \frac{\partial\Gamma^i_{\alpha k}}{\partial x^j} + \Gamma^\beta_{\alpha j}\Gamma^i_{\beta k} - \Gamma^\beta_{\alpha k}\Gamma^i_{\beta j}$$

Transcription of the expressions for the $B^i_{\alpha\,jk}$ is straightforward,

```
curvature[gamma_,i_,a_,j_,k_]:=Module[{b,dim=Length[gamma]},
        Together[
        D[gamma[[i,a,j]],x[k][s]]-D[gamma[[i,a,k]],x[j][s]]+
        Sum[gamma[[b,a,j]]gamma[[i,b,k]]-
                gamma[[b,a,k]]gamma[[i,b,j]],{b,dim}] ]]
```

and full curvature tensor is returned by

```
curvature[gamma_]:=Module[{dim=Length[gamma]},
        B=Array[r1,{dim,dim,dim,dim}];
        Do[Do[Do[Do[B[[i,a,j,k]]=curvature[gamma,i,a,j,k],
                {i,dim}],{a,dim}],{j,dim}],{k,dim}];B]
```

The covariant form, $R_{hijk} = g_{h\alpha}B^\alpha_{i\ jk}$ is called the Riemann-Christoffel curvature tensor.

Example: Display the curvature tensor for the example metric

```
B = curvature[gamma]
```

returns

```
                                2              2
                                c              c
{{{{0, 0}, {0, 0}}, {{0, -------------},{-(-------------), 0}}},
                           2          2  2          2
                           c  + x[1][s]   c  + x[1][s]

         2                 2
         c                 c
{{{0, -(---------------)}, {---------------, 0}},{{0, 0}, {0, 0}}}}
        2           2 2      2           2 2
      (c  + x[1][s] )      (c  + x[1][s] )
```

The Ricci Tensor

The Ricci tensor plays an important role in relativity theory. It is defined as a contraction of the curvature tensor,

$$R_{ij} = B^{a}_{i a j} = \frac{\partial \Gamma^{\alpha}_{i\alpha}}{\partial x^{j}} - \frac{\partial \Gamma^{\alpha}_{ij}}{\partial x^{\alpha}} + \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{\beta j} - \Gamma^{\beta}_{ij}\Gamma^{\alpha}_{\beta\alpha}$$

Thus, to Print[] the components of $R_{ij} = B^{\alpha}_{i\ \alpha j}$, enter the *Mathematica* statement

```
Do[Do[Print["R(",i,",",j,") = ",Simplify[ Sum[B[[a,i,a,j]],{a,2}] ] ],
        {i,2}],       {j,2}]
```

which yields

```
                   2
                   c
      R(1,1) = ---------------
                 2          2 2
               (c  + x[1][s] )
      R(2,1) = 0
      R(1,2) = 0
                  2
                  c
      R(2,2) = ------------
                 2          2
                 c  + x[1][s]
```

N.B., Sum[*expression*,{i,n}] returns the sum of *expression* evaluated for i-values running from 1 to (the integer) n. Do[*expression*,{i,n}] works similarly.

We also transcribe separate *Mathematica* modules to compute the Ricci tensor,

```
Ricci[gamma_,i_,j_]:=Module[{m,n,dim=Length[gamma]},
Together[Sum[D[gamma[[m,i,m]],x[j][s]]-D[gamma[[m,i,j]],x[m][s]]+
        Sum[gamma[[n,i,m]]gamma[[m,n,j]]-
                gamma[[n,i,j]]gamma[[m,n,m]],{n,dim}],{m,dim}]]]
```

and

```
        Ricci[gamma_]:=Module[{dim=Length[gamma]},
                R=Array[r1,{dim,dim}];
                Do[Do[R[[i,j]]=Ricci[gamma,i,j],{i,dim}],{j,dim}];R]
```

Example: Direct computation of Ricci tensor for the example metric.

```
        Ricci[gamma]
```

returns

```
                 2                      2
                c                      c
        {{---------------, 0}, {0, -------------}}
           2          2 2          2          2
         (c  + x[1][s] )         c  + x[1][s]
```

which is identical to the results obtained by contraction of B, as expected.

Exercise: Compute the Christoffel symbols and Ricci tensor for

$$\|g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

PART II. APPLICATION: THE SCHWARZCHILD METRIC .

Development of a spherically-symmetric $\|g_{ij}\|$ consistent with the Einstein equations. Assumption: $\|g_{ij}\|$ is spherically-symmetric. Assume that

$$g = \|g_{ij}\| = \begin{pmatrix} L[r] & 0 & 0 & 0 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2(\theta) & 0 \\ 0 & 0 & 0 & M[r] \end{pmatrix}$$

$$= \begin{pmatrix} L[x[1][s]] & 0 & 0 & 0 \\ 0 & -x[1][s]^2 & 0 & 0 \\ 0 & 0 & -x[1][s]^2\sin^2(x[2][s]) & 0 \\ 0 & 0 & 0 & M[x[1][s]] \end{pmatrix}$$

where the functions L[r] and M[r] are to be determined. Encode via,.

```
g=DiagonalMatrix[{L[x[1][s]],-x[1][s]^2,-(x[1][s]Sin[x[2][s]])^2,M[x[1][s]]}];
```

Conventional Notation

Direct substitution is achieved in *Mathematica*, via the construction

expression/.u->v

which returns *expression* with occurrences of u replaced by v. Thus, to present output in {r,q,j,t}-form, define the following replacement rules:

```
rt={x[1][s]->r,x[2][s]->q,x[3][s]->j,x[4][s]->t};
```

N.B., *Mathematica* 2.1 does *not* have Greek fonts. We substitute j for phi and q for theta here and in the sequel with a text editor.

Compute the Christoffel symbols, etc.

```
setup[g];
```

The Einstein Equations

The Einstein equations require that $R_{ij} = 0$. Compute and display the Ricci tensor.

R=Simplify[Ricci[gamma]]/.rt

returns the Ricci tensor for the spherically-symmetric metric function,

$$\{\{-\left(\frac{L'[r]}{r\,L[r]}\right) - \frac{L'[r]\,M'[r]}{4\,L[r]\,M[r]} - \frac{M'[r]^2}{4\,M[r]^2} + \frac{M''[r]}{2\,M[r]}, 0, 0, 0\},$$

$$\{0, -1 - \frac{1}{L[r]} + \frac{r\,L'[r]}{2\,L[r]^2} - \frac{r\,M'[r]}{2\,L[r]\,M[r]}, 0, 0\},$$

$$\{0, 0, (\mathrm{Sin}[T]^2\,(-2\,L[r]\,M[r] - 2\,L[r]^2\,M[r] + r\,M[r]\,L'[r] - r\,L[r]\,M'[r]))\,/\,(2\,L[r]^2\,M[r])\,, 0\},$$

$$\{0, 0, 0, \frac{M'[r]}{r\,L[r]} - \frac{L'[r]\,M'[r]}{4\,L[r]^2} - \frac{M'[r]^2}{4\,L[r]\,M[r]} + \frac{M''[r]}{2\,L[r]}\}\}$$

Solution of the Einstein Equations

We seek functions L[r] and M[r] such that the Einstein equations are satisfied.

Step 1. Eliminate the M''[r] terms from the 1,1 and 4,4 parts of R_{ij} = R[[i,j]] = 0.

temp=((Simplify[(R[[1,1]]M[r[s]]-R[[4,4]]L[r[s]])/.rt])==0)

returns

$$-\left(\frac{M[r]\,L'[r] + L[r]\,M'[r]}{r\,L[r]}\right) == 0$$

Step 2. Solve for L[r] in terms of M[r]. Use

$$\|g_{ij}\| \underset{r\to\infty}{\rightarrow} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *Mathematica* operator DSolve[] to obtain Rule0:

```
Rule0=ExpandAll[Flatten[DSolve[
        {L[Infinity]==-1,M[Infinity]==1,temp},L[r],r]]]
```

Mathematica issues a "warning:"

```
Solve::ifun:
  Warning: Inverse functions are being used by Solve, so some solutions may not be found.
```

then returns the solution of interest

```
                  1
{L[r] -> -(-------)}
                M[r]
```

Step 3. *Mathematica* will not apply the rule for L[r] to replace L'[r]. Thus, one specifies a separate rule for L'[r] and defines Rule1 as follows:

```
Rule1=Flatten[{Rule0,Solve[temp,L'[r]]/.Rule0}]/.r->x[1][s]
```

which returns

```
                          1                              M'[x[1][s]]
{L[x[1][s]] -> -(-------------), L'[x[1][s]] -> ------------}
                    M[x[1][s]]                                       2
                                                              M[x[1][s]]
```

N.B., Flatten[{{u},{v}}] returns {u,v}, etc. Thus, the subrules comprising Rule1 will be applied consecutively.

Step 4. Use Rule1 and $R_{22} = 0$ to define the form of M. Express M in a rule (Rule2).

```
Rule2=Flatten[DSolve[
        0==Factor[Expand[R[[2,2]]/.Rule1]]/.rt,M,r] ]
```

which returns the rule governing the function M,

$$\{M \to (1 + \frac{C[1]}{\#1}\ \&)\}$$

where C[1] is a constant of integration. The rule on the function works as expected, i.e.,

```
M[x[1][s]]/.Rule2
```

returns

$$1 + \frac{C[1]}{x[1][s]}$$

The Schwarzchild Metric

Applying Rule1 and Rule2, one obtains the form of the spherically-symmetric metric tensor consistent with R = 0:

```
gw=g/.Rule1/.Rule2/.rt
```

returns

$$\{\{-(\frac{1}{1 + \frac{C[1]}{r}}), 0, 0, 0\}, \{0, -r^2, 0, 0\},$$

$$\{0, 0, -(r^2\ Sin[q]^2), 0\}, \{0, 0, 0, 1 + \frac{C[1]}{r}\}\}$$

As expected, this is the Schwarzchild metric form. The usual form of the metric tensor is obtained by identifying the integration constant C[1] with $-2\ G\ M/c^2$, where G is the gravitational constant, M is the central mass, and c is the speed of light.

Exercise: Demonstrate that R = 0 for Schwarzchild gravitation.

Exercise: The geodesic trajectories are given by solutions of the geodesic equations

$$\frac{d^2x[i][s]}{ds^2} + \Gamma^i_{\alpha\beta} \frac{dx[\alpha][s]}{ds} \frac{dx[\beta][s]}{ds} = 0$$

Derive the geodesic equations for

$$\|g_{ij}\| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

Exercise: Derive the geodesic equations for the "example metric,"

$$\|g_{ij}\| = \begin{Vmatrix} 1 & 0 \\ 0 & x[1][s]^2 + c^2 \end{Vmatrix}$$

Exercise: Planetary orbits.

1. Derive the geodesic equations for the Schwarzchild metric.
2. Demonstrate that if q[0] = p/2 and q'[0] = 0, then q[s] = p/2.
   a. Express the geodesic equations for q[s] = p/2.
3. Show that r[s]^2 j'[s] = h = constant.
4. Show that (1 - (2G M/c^2) / r[s]) t'[s] = K = constant.
5. Use the results of parts 1 through 4 and

$$1 = \frac{ds^2}{ds^2} = g_{ij}\frac{dx[i][s]}{ds}\frac{dx[j][s]}{ds}$$

$$= g_{ij}x[i]'[s]x[j]'[s] \to \sum_{i=1}^{4} g_{ij}x[i]'[s]^2$$

to derive the *Mathematica* expression

```
                  2             2                2
                 K             h        (x[1])'[s]
1 == -------------- - ----------- - ---------------
             C[1]               2            C[1]
        1 + --------   x[1][s]      1 + --------
             x[1][s]                      x[1][s]
```

6. Make the change of variables,

x[1][s]->1/u[x[3][s]],

and employ the condition derived in 3, to establish that

(x[1])'[s] -> -(h u'[x[3][s]])

and derive the equations:

$$1 == -(h^2\ u[phi]^2) + \frac{K^2}{1 + C[1]\ u[phi]} - \frac{h^2\ u'[phi]^2}{1 + C[1]\ u[phi]}$$

$$u'[phi]^2 + u[phi]^2 == -h^{-2} + \frac{K^2}{h^2} - \frac{C[1]\ u[phi]}{h^2} - C[1]\ u[phi]^3$$

and

$$u''[phi] + u''[phi] == -\left(\frac{C[1]}{2\ h^2}\right) - \frac{3\ C[1]\ u[phi]^2}{2}$$

The last equation for u[phi] (= 1/r[j]) frequently serves as the basis for a discussion of the advance of the perihelion of planetary orbits. See, for example, Crandall (ref 4). How should this approach be modified to treat photon trajectories?

Exercise: Compute the Christoffel symbols, curvature tensor, Ricci tensor, and geodesic equations for charge-free (i.e., Q = 0) Kerr-Newman gravitation. Demonstrate that R = 0 for the Kerr-Newman metric.

REFERENCES

1. Keith Andrew and Charles G. Fleming, *Comp. in Phys*, Vol. 6, 1992, p. 498.

2. Stephen Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, Addison-Wesley, Redwood City, CA, 1991.

3. The expressions presented in this report may be found in any textbook that addresses Riemann geometry. See, for example, Harry Lass, *Vector and Tensor Analysis*, The Maple Press, York, PA, 1950.

4. Richard E. Crandall, *Mathematica for the Sciences*, Addison-Wesley, Redwood City, CA, 1991.

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NOTE: PLEASE NOTIFY COMMANDER, ARMAMENT RESEARCH, DEVELOPMENT, AND ENGINEERING CENTER, BENÉT LABORATORIES, CCAC, U.S. ARMY TANK-AUTOMOTIVE AND ARMAMENTS COMMAND, AMSTA-AR-CCB-O, WATERVLIET, NY 12189-4050 OF ADDRESS CHANGES.